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VARIATIONAL FORMULATION OF THE PROBLEM OF RETAINED VISCOPLASTIC OIL*

V.M. ENTOV and S.V. PAN'KO

A variational formulation is given for the problem of determining the limiting equilibrium of retained viscoplastic oil during its displacement with water from a stratified bed. It is shown that the basic approximation of the formulation admitting of an effective solution by the methods of the plane problem of non-linear filtering /1–5/ follows naturally from the variational formulation proposed, provided that the class of functions in which the solution is sought is restricted. Some estimates of the volume of the bed from which the oil is displaced are obtained on the basis of the variational formulation.

1. We consider the problem of displacing viscoplastic oil with water from an inhomogeneous stratified bed of constant thickness H, whose permeability $k(z)$ is a monotonically decreasing function of the z coordinate measured from the bottom towards the top of the bed (both the bottom and top of the bed are impermeable). We shall assume that the region D under consideration is bounded by a cylindrical surface Σ perpendicular to the plane $z=0$. The pressure p, independent of z, $p=P(x, y)$, is specified on the part Σ_p of the surface (the feed surface) which is, generally speaking, multiconnected, and the remaining part of the boundary Σ_q is impermeable. At the final stage only water moves within the bed, having displaced the oil from wherever the water pressure gradient exceeds the local value of the limiting gradient for the oil $G(z)$. Henceforth, we shall assume that the relation $k(z)G^2(z) = k_0G_0^2 \equiv C = \text{const}$ holds for viscoplastic oils.

When such a formulation corresponding to washing the bed layer by layer is used, the bed is obviously divided into two regions which we shall call the region of flow and the region of retained oil, and characterized by the fact that in the first region $w>0$, $(x, y, z) \in D_1$, while in the second region $w=0$, $(x, y, z)\in D_2$, where $\mathbf{w}$ is the filtration rate. By virtue of the assumption of a monotonic decrease in the permeability from the bottom towards the top of the bed, the retained oil will be distributed above the zone of flow along each vertical. We shall wite the equation of the interface between the stream of water and retained oil in the form $z=h(x, y)$, putting formally $h=0$ at the points where the retained oil occupies the whole thickness of the bed and $h=H$ the points where no retained oil is present and the whole thickness is occupied only by the flow of water. The function $h(x, y)$ is uniquely defined in a plane region Δ cut by the surface Σ in the plane $z=0$. The region Δ separates into three, mutually non-intersecting subregions

$$\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$$
$$h = H\ (x, y) \in \Delta_1;\ 0 < h < H\ (x, y) \in \Delta_2;\ h = 0\ (x, y) \in \Delta_3$$

The equations of motion have, for the present scheme, the form /1/

$$\mathbf{w} = -(k(z)/\mu)\nabla p,\ |\nabla p| > G(z),\ (x, y, z) \in D_1;\ \operatorname{div} \mathbf{w} = 0 \tag{1.1}$$
$$w \equiv 0,\ |\nabla p| \leqslant G(z),\ (x, y, z) \in D_2,\ \mu = \text{const}$$

Therefore the problem reduces to that of finding a solution of the equation

$$\operatorname{div}(k(z) \operatorname{grad} p) = 0 \tag{1.2}$$

in the region of water flow D_1, satisfying the condition of impermeability at the parts of the bottom and top adjacent to the region of flow at the boundary Σ_q:

$$\partial p/\partial z = 0,\ z = 0\ (x, y) \in \Delta_1 \cup \Delta_2;\ z = H\ (x, y) \in {}_1 \tag{1.3}$$
$$\partial p/\partial n = 0 \quad (x, y, z) \in \Sigma_q$$

and the conditions at the unknown oil-water interface

$$\partial p/\partial n = 0,\ |\nabla p| = G(z);\ z = h(x, y),\ (x, y) \in \Delta \tag{1.4}$$

In addition, the solution must take known values at the feed surface Σ_p.

We shall show that the initial problem admits of the variational formulation. Let us consider the functional

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$$J = \frac{1}{2} \iint\limits_{\Delta} \int\limits_{0}^{h(x,y)} [k(z) |\nabla p|^2 - k_0 G_0^2] \, dz \, dx \, dy \tag{1.5}$$

and compute its first variation when $p(x, y, z)$ and $h(x, y)$ are varied. We can do this in two ways. If we assume that the function $p(x, y, z)$ is defined over the whole layer $\Delta \times [0, H]$ and is continued smoothly to the region containing the retained oil $z > h(x, y)$, then for **fixed** $p(x, y, z)$ we are interested in varying the upper limit of the integral and obviously have

$$\delta J = \frac{1}{2} \iint\limits_{\Delta_2} [k(z) |\nabla p|^2 - k_0 G_0^2]_{z=h(x,y)} \delta h \, dx \, dy - \tag{1.6}$$

$$\iint\limits_{\Delta} \int\limits_{0}^{h(x,y)} \nabla (k(z) \nabla p) \delta p \, dz \, dx \, dy + \iint\limits_{S} k(z) \frac{\partial p}{\partial n} \delta p \, dS$$

where δp and δh are independent variations.

We can, however, assume that the function $p(x, y, z)$ is defined only for $0 \leqslant z \leqslant h(x, y)$. Then, varying $h(x, y)$ is invariably connected with varying $p(x, y, z)$, and using the formula for varying the integral with variable integration limits /6/ we have

$$\delta J = -\iint\limits_{\Delta} \int\limits_{0}^{h} \nabla (k \nabla p) \delta^* p \, dz \, dx \, dy + \iint\limits_{S} k (\nabla p \mathbf{n}) \delta p \, dS + \frac{1}{2} \iint\limits_{S} [k(z) |\nabla p|^2 - C] \, \delta \mathbf{x} \cdot \mathbf{n} \, dS \tag{1.6'}$$

Here $\delta^* p$ and $\delta \mathbf{x}$ are independent scalar and vector variations defined in D, and S is the surface forming the boundary of the region of water flow and consisting of parts of the surface Σ, sections of the top and bottom of the bed, and of the boundary between the retained oil and the region of water flow. Assuming that the variation $\delta \mathbf{x}$ (displacement of a point of the region") is zero at the boundary of the region D everywhere except at the retained oil surface $z = h(x, y)$ where $\delta \mathbf{x} = \mathbf{k} \delta h$, we can confirm that relations (1.6) and (1.6') are identical

Clearly, if a pair of functions $\{h(x, y),\ p(x, y, z)\}$; $(x, y) \in \Delta,\ 0 \leqslant z \leqslant h$ gives a solution to the problem of determining the retained oil (1.1)–(1.4), then the variation δJ vanishes. Conversely, from the demand that the variation δJ should vanish when the boundary δh and pressure δp are varied arbitrarily, we have

$$\nabla(k(z) \nabla p) = 0, \ (x, y, z) \in D_1 \tag{1.7}$$

$$|\nabla p| = G(h), \ \partial p / \partial n = 0, \ 0 < z = h(x, y) < H(x, y) \in \Delta_2$$

$$\partial p / \partial z = 0, \ z = 0, \ (x, y) \in \Delta_1 \cup \Delta_2; \ z = H, \ (x, y) \in \Delta_1$$

i.e. the problem which was formulated above.

If the boundary $h(x, y)$ of the retained oil is fixed, then the functional transforms into the additional total potential of dissipation of filtration flow, and the function $p(x, y, z)$ which is a solution of problem (1.7) with condition $|\nabla p|_h = G(h)$ removed, imparts to it a minimum (see e.g. /5/). We shall denote this function by p_h. The functional J transforms, in the class of functions p_h, into a functional of $h(x, y)$

$$J[p_h, h] = J^*[h] = \tag{1.8}$$

$$\frac{1}{2} \iint\limits_{\Delta} \int\limits_{0}^{h} k |\nabla p|^2 \, dz \, dx \, dy - \frac{1}{2} \iint\limits_{\Delta} \int\limits_{0}^{h} k_0 G_0^2 \, dz \, dx \, dy$$

If the surface Σ_p can be separated into two parts, Σ_p^+ and Σ_p^-, denoting the entry and exit of the filtration flow at which the pressure takes constant values of p^+ and p^-, $p^+ > p^-$ respectively, then the first integral in formula (1.8) will be equal to $Q(p^+ - p^-)/2$ where Q is the filtration flow intensity. In general, it is equal to half of the total intensity N dissipated by the filtration flow (here $\mu = 1$).

The second term of the formula (1.8) is equal to $\frac{1}{2} k_0 G_0^2 V_+$ where V_+ is the volume of the region occupied by the moving water. Thus we have

$$J^*[h] = \frac{1}{2} N(h) - \frac{1}{2} C V_+(h), \ C = k_0 G_0^2 \tag{1.9}$$

From the general properties of the linear filtration flows it follows that the functional N depends monotonically on h:

$$N[h^+] \geqslant N[h^-], \ h^+(x, y) \geqslant h^-(x, y)$$

(for the given values of the pressure at the feed surface, the total dissipation of the filtration flow increases, provided that the region of filtration expands due to removal of the impermeable boundary /1, 5/. Clearly, the functional $V_+[h]$ is also monotonic.

When $h(x, y)$ is varied, the variation δJ^* consists of the variation directly related to the deformation of the boundary, and the variation caused by the change in the field p_h. However, since the field p_h is itself a solution of the variational problem, it follows that the corresponding first variation becomes zero. We have here

$$\delta J^* = \frac{1}{2} \iint\limits_{\Delta_2} k(|\nabla p_h|^2 - G^2)_{z=h}\, \delta h\, dx\, dy =$$

$$\frac{1}{2} \iint\limits_{\Delta_2} k\, |\nabla p|^2_{z=h} \delta h\, dx\, dy - \frac{1}{2} C \iint\limits_{\Delta_2} \delta h\, dx\, dy$$

The condition that the variation δJ^* should vanish again yields the following additional boundary condition:

$$|\nabla p_h| = G(h), \quad z = h(x, y)$$

which is used to determine the unknown boundary.

It can be shown that the solution $h(x, y)$ sought furnishes the functional $J^*[h]$ with a maximum on the functions p_h, in any case, when the surfaces $h(x, y)$ are sufficiently close.

Indeed, let the pair $\{p_0(x, y, z), h_0(x, y)\}$ be a solution of the problem. We shall consider the variation of the retained oil boundary, assuming that

$$h = h_0(x, y) + \varepsilon\eta(x, y), \quad \varepsilon \geqslant 0$$

We take the field

$$p_h^* = p_0(x, y, z), \quad 0 \leqslant z \leqslant \min\{h_0, h\}$$
$$p_h^* = p_0(x, y, h_0), \quad h_0 \leqslant z \leqslant h_0 + \varepsilon\eta$$

as the trial pressure field in the deformed region D_h. Let p_h be the true pressure field for the deformed region. The general property of minimality of the supplementary dissipation potential on the solutions implies that

$$\iiint\limits_{D_h} k(z)\, |\nabla p_h|^2\, dV \leqslant \iiint\limits_{D_h} k(z)\, |\nabla p_h^*|^2\, dV \tag{1.10}$$

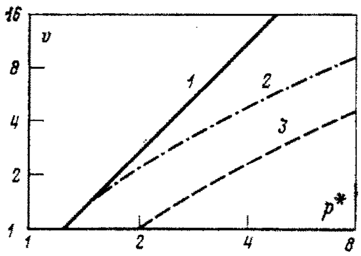
On the other hand, for the trial field we have

$$J[p_h^*, h] - J[p_0, h_0] = \iint\limits_{\Delta_2^+} \int\limits_{h_0}^{h_0+\varepsilon\eta} k(z)\, [|\nabla p_h^*|^2 - G^2(z)]\, dz - \tag{1.11}$$

$$\iint\limits_{\Delta_2^-} \Big[\int\limits_{h_0+\varepsilon\eta}^{h_0} k(z)\, [|\nabla p_0|^2 - G^2(z)]\, dz \leqslant \iint\limits_{\Delta_2^+} k(z)\, [|\nabla p_h^*|^2 - G^2(z)]\, dz$$

Here $\Delta_2^{\pm}$ are the subregions of Δ_2 in which the variation η has the corresponding sign. In the layer $h_0 \leqslant z \leqslant h_0 + \varepsilon\eta\, |\nabla p_h^*| \leqslant |\nabla p_0(x, y, h_0)| \leqslant G(h_0) \leqslant G(z)$, so that theintegral on the right-hand side of the inequality (1.11) is non-positive. Therefore, taking into account (1.10) and (1.11), we have

$$J^*[h] \leqslant J[p_h^*, h] \leqslant J[p_{h_0}, h_0]$$



Thus the solution of the initial problem reduces to the following minimax problem: to find a function $h(x, y)$ such that the minimum of the integral (1.4) over all admissible $p(x, y, z)$ takes its maximum value.

We note that expression (1.9), taking the positiveness of the integrand in (1.8) into account, yields

$$V_+[h] \leqslant C^{-1} N[h] \leqslant C^{-1} N[H] \tag{1.12}$$

Here $N[H]$ is the dissipation intensity for the layer in which water moves along the whole layer thickness ($h \equiv H$). Since its value is found from the solution of the standard linear filtration problem, it follows that the simple estimate (1.12) may turn out to be very useful. For example, for a flow in a borehole of radius ρ measured from the rectilinear feed contour we have

$$Q_0 = \frac{2\pi k^0 H}{\mu} \frac{p^+ - p^-}{\ln(2a/\rho)}, \qquad N = \frac{2\pi k^0 H}{\mu} \frac{(p^+ - p^-)^2}{\ln(2a/\rho)}$$

where k^0 is the mean layer permeability, and a is the distance between the feed contour and the borehole. Then, by virtue of (1.12) we have

$$V_+ \leqslant \frac{\mu N}{k_0 G_0^2} \leqslant \frac{2\pi k^0}{k_0} \frac{Ha^2}{\ln(2a/\rho)} \left(\frac{p^+ - p^-}{aG}\right)^2 \tag{1.13}$$

The straight line 1 in the figure corresponds to the estimate (1.13) (for a homogeneous layer $k^0 = k_0$ and $a/\rho = 10^3$) We shall see later that the estimate is very approximate.

2. Consider the minimax problem formulated above, in the class of pressure fields independent of the coordinate z, $p = p(x, y)$. Then we have

$$J = \frac{1}{2}\iint\limits_{\Delta} (\nabla p(x, y))^2 \int\limits_0^h k(z)\,dz\,dx\,dy - \frac{1}{2} C \iint\limits_{\Delta} h\,dx\,dy \tag{2.1}$$

The requirement that J be minimal for fixed $h(x, y)$ gives

$$\nabla (K(h) \nabla p(x, y)) = 0;\; p(x, y) = f,\; (x, y) \in \Sigma_p \tag{2.2}$$
$$\partial p / \partial n = 0, \quad (x, y) \in \Sigma_q; \quad K(h) = \int\limits_0^h k(z)\,dz$$

Maximizing now the functional (2.1) with respect to h taking (2.2) into account, we obtain

$$\delta J = \frac{1}{2}\iint\limits_{\Delta} [K(h) |\nabla p|^2 - k(h) G^2(h)]\,\delta h\,dx\,dy \tag{2.3}$$

The variation δh is arbitrary for $0 < h < H$; $\delta h > 0$ when $h = 0$, $\delta h < 0$ when $h = H$. Thus the condition that J is maximum yields

$$|\nabla p|^2 = G^2(h),\; 0 < h < H;\; K(h)(|\nabla p|^2 - G^2(h)) \leqslant 0, \tag{2.4}$$
$$h = 0$$
$$|\nabla p|^2 - G^2(h) \geqslant 0,\; h = H$$

The problem (2.3), (2.4) is identical with the problem of finding the retained oil in stratified inhomogeneous beds under the assumption that the pressure is distributed hydrostatically throughout the layer thickness (the intercalations are in perfect contact) formulated in /1/. The problem reduces to the plane problem of non-linear filtration and, in a number of cases, has an effective soltuion /1, 2/. The arguments given show that the corresponding solution (h^*, p^*) gives the functional J a maximum in the class of functions p independent of z. Thus

$$J_0 = \max_h \min_p J \leqslant J^* = J[h^*, p^*]$$

3. Next we shall consider the functional J defined by (1.5) in the class of step functions $h(x, y)$ taking only two different values

$$h = 0,\; (x, y) \in \Delta_3 = \Delta \setminus \Delta_1;\; h = H,\; (x, y) \in \Delta_1 \tag{3.1}$$

Here the part Γ_0 of the flow region boundary Δ_1 coincides with the feed contour Γ_p and the fixed impermeable boundary Γ_q, and the part Γ_1 is unknown. We have

$$J = J_1 = \frac{1}{2}\iint\limits_{\Delta_1}\int\limits_0^H [k(z) |\nabla p|^2 - C]\,dz\,dx\,dy \tag{3.2}$$

Constructing the variation δJ_1 we obtain

$$\delta J_1 = -\iint\limits_{\Delta_1}\int\limits_0^H \nabla (k(z) \nabla p)\,\delta^* p\,dz\,dx\,dy +$$
$$\int\limits_{\Gamma_0 \cup \Gamma_1}\int\limits_0^H k(z) \frac{\partial p}{\partial n}\,\delta^* p\,dz\,dl + \frac{1}{2}\int\limits_{\Gamma_1} \mathbf{n}\delta\mathbf{x} \int\limits_0^H [k(z) |\nabla p|^2 - C]\,dz\,dl$$

Equating the variation δJ_1 to zero, we have

$$\int\limits_0^H \nabla (k(z) \nabla p)\,dz = 0, \quad (x, y) \in \Delta_1; \quad p = f(x, y), \quad (x, y) \in \Gamma_p \tag{3.3}$$
$$\int\limits_0^H k(z) [|\nabla p|^2 - G^2]\,dz = 0, \quad \frac{\partial p}{\partial n} = 0, \quad (x, y) \in \Gamma_1 \cup \Gamma_q$$

If the pressure specified on the feed contour is independent of z, then problem (3.3) admits of the solution $p = p(x, y)$, and as a result we arrive at the boundary value problem

$$\Delta p = 0,\; (x, y) \in \Delta_1;\; p(x, y) = f(x, y),\; (x, y) \in \Gamma_p \tag{3.4}$$
$$\partial p / \partial n = 0,\; (x, y) \in \Gamma_q'$$
$$|\nabla p|^2 = C/k^\circ,\; \partial p / \partial n = 0,\; (x, y) \in \Gamma_1$$

Here we have the well-known plane formulation of the problem on retained oil in homogeneous layers /3, 4, 6/, which can be solved efficiently using the jet theory. Since the class of admissible functions $h(x, y)$ is restricted in the corresponding variational formulation, it follows that the solutions obtained yield lower estimates for the functional J on the "true solution"

$$J \geqslant J_1 \tag{3.5}$$

Relation (3.5) can also be used to estimate the unknown values of the flushed volume of the layer V_+ as was done in Sect.1. Using (1.9) and (3.5) we obtain

$$V_+ = (N - 2J)\, C^{-1} \leqslant C^{-1} (N_D - 2J_1) = C^{-1} (N_D - N_1) + V_1 = V^* \qquad (3.6)$$

Here N_D denotes the efficiency with the same geometry, a given pressure drop and motion following Darcy's law $(G \equiv 0)$, N_1 is the dissipation efficiency for a flow with the formation of retianed oil blocks, in the framework of the plane problem (3.4), V_1 is the volume of the flushed zone computed under the same assumptions as in the case of the plane problem.

Since a large reserve of available solutions exists corresponding to the two-dimensional formulation (3.4) /3, 4, 7, 8/, it follows that N_1 and V_1 can be easily computed in explicit form. Thus, the required solution for the flow discussed above, moving from a single borehole to the rectilinear feed contour was given in /7, 8/. The relation $V^* (\Delta p/aG)$ computed with help of this solution is depicted by curve *2* in logarithmic coordinates.

According to (3.6) the relation yields an upper estimate for the volume of the flushed zone. In the present case it can be directly confirmed how approximate the estimate (3.6) is. The crux of the matter is that the example chosen refers to the class of flows for which the plane formulation yields an exact solution of the three-dimensional problem /1/. For this reason it is easy, in this case, to compute the exact volume of the flushed zone V_+ using the same, already constructed solution /7, 8/. Curve *3* in the figure shows the corresponding result. We see that in the basic domain of parameter variation the estimate (3.6) exceeds the exact value by a factor of approximately 2.

Let us now assume that the step function $h(x, y)$ can take the $n + 1$-th given discrete value

$$h_0 = 0 < h_1 < \ldots < h_j < \ldots < h_n = H$$
$$h(x, y) = h_j, \quad (x, y) \in \Delta_j$$

Then the boundaries Γ_j between the regions Δ_{j+1} and Δ_j and the continuous pressure field $p(x, y, z)$ will have to be determined. We shall restrict ourselves to two-dimensional fields $p(x, y)$. Requiring a minimum of the functional

$$J = J_n = \frac{1}{2} \iint\limits_{\Delta_j} \int\limits_0^{h_j} [k(z) |\nabla p|^2 - C]\, dz\, dx\, dy$$

in p and a maximum in the possible configurations of the boundaries Γ_j, we arrive at the problem

$$\partial^2 p_j/\partial x^2 + \partial^2 p_j/\partial y^2 = 0, \ (x, y) \in \Delta_j$$
$$p_j = p_{j+1}, \quad k_j^* \frac{\partial p_j}{\partial n} = k_{j+1}^* \frac{\partial p_{j+1}}{\partial n}, \quad (x, y) \in \Gamma_j,$$
$$k_j^* = \int\limits_0^{h_j} k(z)\, dz$$
$$k_{j+1}^* \left[\left(\frac{\partial p_{j+1}}{\partial s} \right)^2 - \left(\frac{\partial p_{j+1}}{\partial n} \right)^2 \right] - k_j^* \left[\left(\frac{\partial p_j}{\partial s} \right)^2 - \left(\frac{\partial p_j}{\partial n} \right)^2 \right] =$$
$$C(h_{j+1} - h_j), (x, y) \in \Gamma_j$$
$$p = f(x, y), \ (x, y) \in \Gamma_p; \ \partial p/\partial n = 0, \ (x, y) \in \Gamma_q$$

The case of a two-dimensional pressure field and a step function $h(x, y)$ considered, corresponds to the appropriate formulation of the problem on retained oil in stratified bed disucssed /9/. However, the condition of limit equilibrium at the boundaries Γ_j obtained above from the general variational formulation, differs from that formulated in /9/ and based on physical considerations.

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ON THE JOINT APPLICATION OF CARTESIAN AND BIPOLAR COORDINATES TO SOLVE BOUNDARY VALUE PROBLEMS OF POTENTIAL THEORY AND ELASTICITY THEORY*

V.S. PROTSENKO and A.I. SOLOV'EV

Equations are obtained that connect harmonic functions with separated variables in Cartesian and bipolar coordinates. These equations can be used to investigate a number of new boundary value problems of potential theory and elasticity theory for domains bounded by Cartesian and bipolar coordinate system coordinate lines.

1. Consider a plane domain whose boundary is formed by two intersecting circles. The solution of internal boundary value problems for such domains (circular cresents) is found in bipolar coordinates α, β defined by the relations $(a > 0)$ [1]

$$x = \frac{a \operatorname{sh} \alpha}{\operatorname{ch} \alpha + \cos \beta}, \quad y = \frac{a \sin \beta}{\operatorname{ch} \alpha + \cos \beta} \quad (-\infty < \alpha < \infty, -\pi \leqslant \beta \leqslant \pi) \tag{1.1}$$

The arcs of the circles forming the circular cresent are the coordinate lines $\beta = \text{const}$, and pass through the point $x = \pm a$, $y = 0$. The quantity β is measured by the angle between the tangent to the arc at the point $x = a$, $y = 0$ and the segment $(-a, a)$ of the x axis corresponding to the value $\beta = 0$. Within the domain under consideration the coordinate α varies between the limits $-\infty$ and ∞. Particular solutions of the Laplace equation in bipolar coordinates, obtained by separation of variables and bounded as $\alpha \to \pm\infty$, have the following form

$$\cos \lambda\alpha \left\| \begin{matrix} \operatorname{ch} \lambda\beta \\ \operatorname{sh} \lambda\beta \end{matrix} \right\|, \quad \sin \lambda\alpha \left\| \begin{matrix} \operatorname{ch} \lambda\beta \\ \operatorname{sh} \lambda\beta \end{matrix} \right\| \quad (-\infty < \lambda < \infty)$$

Theorem 1. The following equations hold for $-\pi < \beta < \pi$

$$\operatorname{sh} \lambda y \left\| \begin{matrix} \cos \lambda x \\ \sin \lambda x \end{matrix} \right\| = \int_{-\infty}^{\infty} C(\lambda, \tau) \operatorname{sh} \tau\beta \left\| \begin{matrix} \cos \tau\alpha \\ \sin \tau\alpha \end{matrix} \right\| d\tau \tag{1.2}$$

$$\operatorname{ch} \lambda y \left\| \begin{matrix} \cos \lambda x \\ \sin \lambda x \end{matrix} \right\| - \left\| \begin{matrix} \cos \lambda a \\ 0 \end{matrix} \right\| = \int_{-\infty}^{\infty} C(\lambda, \tau) \operatorname{ch} \tau\beta \left\| \begin{matrix} \cos \tau\alpha \\ \sin \tau\alpha \end{matrix} \right\| d\tau$$

$$C(\lambda, \tau) = \frac{\lambda a}{\operatorname{sh} \pi\tau} e^{-i\lambda a} \Phi(1 - i\tau, 2; 2i\lambda a) \equiv \frac{\lambda a}{\operatorname{sh} \pi\tau} e^{i\lambda a} \Phi(1 + i\tau, 2; -2i\lambda a)$$

The last identity follows from the Kummer transformation /2, 3/ for the degenerate hypergeometric function.

The boundary value problems for a cresent domain containing an infinitely remote point are solved conveniently in bipolar coordinates α, σ

$$x = \frac{a \operatorname{sh} \alpha}{\operatorname{ch} \alpha - \cos \sigma}, \quad y = \frac{a \sin \sigma}{\operatorname{ch} \alpha - \cos \sigma}$$
$$(-\infty < \alpha < \infty, -\pi \leqslant \sigma \leqslant \pi, a > 0)$$

The quantity σ is measured by the angle between the tangent to the arc at the point $x = a$, $y = 0$ and the ray (a, ∞) on the x axis corresponding to the value $\sigma = 0$.

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